Analyticity and regularity for a class of second order evolution equations.

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ANALYTICITY AND REGULARITY FOR A CLASS OF SECOND ORDER EVOLUTION EQUATIONS

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Résumé. On étudie la conservation de la régularité et l'effet régularisant pour l'équation $u'' + Au + cA^{\alpha}u' = 0$ où A est un opérateur auto-adjoint positif sur un espace de Hilbert réel H et $\alpha \in (0, 1]$; $c > 0$. Si $\alpha \ge \frac{1}{2}$ l'équation engendre un semi-groupe analytique sur $D(A^{1/2}) \times H$, et si $\alpha \in (0, \frac{1}{2})$ une propriété de régularisation plus faible mais optimale est prouvée. Enfin des propriétés de conservation de la régularité associée à d'autres normes sont obtenues, avec comme exemple typique d'application l'équation des ondes $u_{tt} - \Delta u - c\Delta u_t = 0$ avec condition de Dirichlet homogène dans un domaine borné pour laquelle la régularité $C_0(\Omega) \times C_0(\Omega)$ est conservée for $t > 0$, un saisissant contraste avec le cas conservatif $u_{tt} - \Delta u = 0$ dans lequel la régularité $C_0(\Omega)$ peut-être perdue même pour un état initial $(u_0, 0)$ avec $u_0 \in C_0(\Omega) \cap C^1(\overline{\Omega})$.

Abstract. The regularity conservation as well as the smoothing effect are studied for the equation $u'' + Au + cA^{\alpha}u' = 0$ where A is a positive selfadjoint operator on a real Hilbert space H and $\alpha \in (0, 1]$; $c > 0$. When $\alpha \ge \frac{1}{2}$ the equation generates an analytic semigroup on $D(A^{1/2}) \times H$, and if $\alpha \in (0, \frac{1}{2})$ a weaker optimal smoothing property is established. Some conservation properties in other norms are established, a typical example being the strongly dissipative wave equation $u_{tt} - \Delta u - c\Delta u_t = 0$ with Dirichlet boundary conditions in a bounded domain for which the space $C_0(\Omega) \times C_0(\Omega)$ is conserved for $t > 0$, in sharp contrast with the conservative case $u_{tt} - \Delta u = 0$ for which $C_0(\Omega)$ -regularity can be lost even starting from an initial state $(u_0, 0)$ with $u_0 \in C_0(\Omega) \cap C^1(\overline{\Omega})$.

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1. Introduction and notation

This paper is mainly devoted to a detailed study of the regularity conservation as well as the smoothing effect for the equation

$$u'' + Au + cA^{\alpha}u' = 0 \tag{1. 1}$$

where A is a positive selfadjoint operator on a real Hilbert space H and $\alpha \geq 0, \quad c > 0$. Throughout the text we assume

$$A \geq \eta I, \quad \eta > 0.$$

In particular the operator A^s is well defined and bounded $H \to H$ for all $s \leq 0$. We identify H with its topological dual and we therefore have

$$D(A) \subset V = D(A^{1/2}) \subset H = H' \subset V' \subset D(A)'.$$

More generally we define a monotone nonincreasing one-parameter family of Hilbert spaces by the formula

$$H^s = \begin{cases} D(A^{s/2}) & \text{if} \quad s \geq 0 \\ (D(A^{-s/2}))' & \text{if} \quad s < 0. \end{cases}$$

We are interested in the smoothing effect and the conservation of regularity for the evolution equation (1.1). The plan of the paper is as follows: Section 2 is devoted to well-posedness of (1.1) considered as a first order system in $V \times H$. Section 3 deals with compactness properties of the resolvent and the semi-group associated to (1.1). Sections 4 and 5 are devoted to the study of time and spatial smoothing properties and especially to a simple direct proof of analyticity for $\alpha \geq 1/2$. A more general result, motivated by a conjecture of Chen & Russell [2] can be found in [3] but their proof is quite involved, relying on a stationary type of argument involving complex resolvent. Here we give a pure real and dynamical argument based only on elementary tools such as inner products and integration with respect to t. In Section 6 we investigate some regularity conservation properties which are specific to (1.1) with $c > 0$ since the conservative problem corresponding to $c = 0$ does not satisfy those properties anymore, cf. eg. [5]. Finally Section 5 contains the basic examples of application.

2. The initial value problem

In this section we consider the equation

$$u'' + Au + Bu' = 0 \tag{2. 1}$$

where $B = B^* \geq 0$ on H and in addition for some constants $C > c > 0$

$$cA^{\alpha} \leq B \leq CA^{\alpha}.$$

Theorem 2.1. *Let $(u^0, u^1) \in V \times H$ be given. For any $T > 0$ there is a unique $u \in C([0,T], V) \cap C^1([0,T], H)$ such that, setting $\beta = \max\{1, \alpha\}$ we have*

$$\begin{cases} u' \in L^2(0,T; H^{\alpha}) & u'' \in L^2(0,T; H^{-\beta}) \\ u'' = -Au - Bu' & \textit{in} \quad L^2(0,T; H^{-\beta}) \\ u(0) = u^0 & u'(0) = u^1. \end{cases}$$

Moreover the function

$$E(t) := \frac{1}{2}\{|u(t)|_V^2 + |u'(t)|_H^2\}$$

is absolutely continuous with

$$E'(t) = - < Bu', u' >_{H^{-\alpha}, H^{\alpha}}$$

almost everywhere on $(0, T)$.

Proof. For the existence part we introduce $J_\lambda = (I + \lambda A)^{-1}$ and $K_\lambda = J_\lambda^n$ for some integer $n \geq \max\{1, \alpha\}$. Then we solve

$$u_\lambda'' + Au_\lambda + K_\lambda B K_\lambda u_\lambda' = 0; \qquad u_\lambda(0) = u^0, \quad u_\lambda'(0) = u^1.$$

The identity

$$\int_0^T (BK_\lambda u_\lambda', K_\lambda u_\lambda')dt + \frac{1}{2}\{\|u_\lambda'(T)\|_H^2 + \|u_\lambda(T)\|_V^2\} = \frac{1}{2}\{\|u^1\|_H^2 + \|u^0\|_V^2\}$$

allows to pass to the limit as $\lambda \to 0$ along a suitable subsequence.

For the uniqueness part as well as the energy identity we start with a solution $u \in L^2(0, T; V) \cap H^1(0, T; H)$ of

$$u'' + Au = f \in L^2(0, T; H^{-\beta})$$

We show that $v_\lambda = K_\lambda u$ satisfies

$$\frac{d}{dt}\frac{1}{2}\{\|v_\lambda'(t)\|_H^2 + \|v_\lambda(t)\|_V^2\} = (K_\lambda f, K_\lambda u') =< f, K_\lambda^2 u' >$$

Then we integrate and let $\lambda \to 0$. Finally we choose $f = -Bu'$. Uniqueness follows then by linearity from the energy identity applied with $u^0 = u^1 = 0$. $\square$

3. Compactness of the resolvent

In this section we assume $B = cA^\alpha, \quad c > 0$. Setting $v = u', \quad U = (u, v) \in V \times H = \mathcal{H}$, and denoting by $\tilde{B}$ the unique extension to $\mathcal{L}(V, H^{1-2\alpha})$ of $B \in \mathcal{L}(D(B), H)$, we find that the equation (2.1) becomes

$$U' + \mathcal{A}U = 0$$

where

$$D(\mathcal{A}) = \{(u, v) \in V \times V, \quad Au + \tilde{B}v \in H\}$$

and

$$\mathcal{A}(u, v) = (-v, Au + \tilde{B}v)$$

so that

$$\|\mathcal{A}(u, v)\|_{\mathcal{H}} \sim \|v\|_V + \|Au + \tilde{B}v\|_H$$

Theorem 3.1. *a) If* $\alpha < 1$, *then* $D(\mathcal{A}) \subset H^\gamma \times V$ *with* $\gamma = \min\{2, 3 - 2\alpha\}$. *As a consequence if the imbedding* $V \to H$ *is compact, then so is* $(I + \mathcal{A})^{-1} : \mathcal{H} \to \mathcal{H}$.

b)If $\alpha \geq 1$, *then we have*

$$\forall z \in V, \quad U(z) =: (cz, -A^{1-\alpha}z) \in D(\mathcal{A})$$

with

$$\|U(z)\|_{\mathcal{H}} + \|\mathcal{A}U(z)\|_{\mathcal{H}} \leq K\|z\|_V.$$

In particular if $\dim H = \infty$, $(I+\mathcal{A})^{-1} : \mathcal{H} \to \mathcal{H}$ *is not compact. Hence in this case the semi-group* $\mathcal{S}(t)$ *generated on* $\mathcal{H}$ *by* $\mathcal{A}$ *is never compact.*

Proof. a) If $U \in D(\mathcal{A})$, then $v \in V$ and $Au + \tilde{B}v \in H$, hence

$$u \in D(A) + A^{-1}(\tilde{B}V) = H^2 + A^{\alpha-1}(H^1) = H^\gamma.$$

If $V \to H$ is compact and $\alpha < 1$, then $H^\gamma \to V$ is compact, therefore since

$$(I+\mathcal{A})^{-1} \in \mathcal{L}(\mathcal{H}, H^\gamma \times \mathcal{V})$$

the result is now obvious.

b) Assume now $\alpha \geq 1$. Then clearly $\forall z \in V, \quad U(z) =: (cz, -A^{1-\alpha}z) \in D(\mathcal{A})$ because both cz and $-A^{1-\alpha}z$ are in V and in addition

$$A(cz) + cA^\alpha(-A^{1-\alpha}z) = 0 \in H.$$

Moreover we have

$$\|U(z)\|_{\mathcal{H}} \leq \|cz\|_V + \|A^{1-\alpha}z\|_H \leq c\|z\|_V + C'\|z\|_H \leq C''\|z\|_V$$

and

$$\|\mathcal{A}U(z)\|_{\mathcal{H}} \leq \|A^{1-\alpha}z\|_V + \|A(cz)+cA^\alpha(-A^{1-\alpha}z)\|_H = \|A^{1-\alpha}z\|_V \leq C'''\|z\|_V.$$

Hence

$$\|U(z)\|_{\mathcal{H}} + \|\mathcal{A}U(z)\|_{\mathcal{H}} \leq K\|z\|_V$$

with $K := C'' + C'''$. Finally when $z \in V$ varies in the unit ball

$$\{\|z\|_V \leq 1\}$$

the first projection of $U(z)$ covers the entire ball of radius c in V, therefore $(I+\mathcal{A})^{-1} : \mathcal{H} \to \mathcal{H}$ is not compact $\square$

4. Analytic type time smoothing effect.

In this section we give a new and short proof of a result previously obtained by Chen & Triggiani [3]. However our proof seems to be limited to the case $B = cA^\alpha, \quad c > 0$ or at least to require that B commutes with A.

Theorem 4.1. *For any* $\alpha \geq 1/2$, *the semi-group* $\mathcal{S}(t)$ *generated on* $\mathcal{H}$ *by* $\mathcal{A}$ *is analytic, more precisely*

$$\forall t > 0, \quad S(t)U^0 \in D(\mathcal{A})$$

and

$$\forall t > 0, \quad \|\mathcal{A}S(t)U^0\|_{\mathcal{H}} \leq \frac{C}{t}\|U^0\|_{\mathcal{H}}. \tag{4. 1}$$

Proof. We set

$$E := \frac{1}{2}\Big(|u'(0)|^2 + |A^{1/2}u(0)|^2\Big) = \frac{1}{2}\|U^0\|^2_{\mathcal{H}}.$$

Multiplying the equation by u' we have immediately

$$\forall t \geq 0, \quad \int_0^t (Bu', u')ds \leq E. \tag{4. 2}$$

In particular, with $C = 1/c$

$$\forall t \geq 0, \quad \int_0^t |A^{\frac{\alpha}{2}}u'(s)|^2 ds \leq CE. \tag{4. 3}$$

Taking the inner product in H of (2.1) by $A^{1-\alpha}u(s)$ and integrating, we find

$$\int_0^t |A^{1-\frac{\alpha}{2}}u(s)|^2 ds = -\int_0^t (u'' + Bu', A^{1-\alpha}u)ds.$$

Now we have

$$\int_0^t (Bu', A^{1-\alpha}u)ds = c\int_0^t (A^\alpha u', A^{1-\alpha}u)ds = c\int_0^t (Au, u')ds$$

$$= \frac{c}{2}(|A^{1/2}u(t)|^2 - |A^{1/2}u(0)|^2) \geq -cE.$$

Next , integrating by parts we find

$$\int_0^t (u'', A^{1-\alpha}u)ds = [(u', A^{1-\alpha}u)]_0^t - \int_0^t (u', A^{1-\alpha}u')ds.$$

Since $\alpha \geq 1/2$, we have $1-\alpha \leq 1/2$ and $1-\alpha \leq \alpha$, hence

$$-\int_0^t (u'', A^{1-\alpha}u)ds \leq C_1 E$$

therefore

$$\int_0^t |A^{1-\frac{\alpha}{2}}u(s)|^2 ds \leq (c + C_1)E = C_2 E \tag{4. 4}$$

hence

$$\int_0^t (|A^{1-\frac{\alpha}{2}}u(s)|^2 + |A^{\frac{\alpha}{2}}u'(s)|^2)ds \leq C_3 E.$$

Since $\frac{1-\alpha}{2} \leq \frac{\alpha}{2}$, we deduce

$$\int_0^t (|A^{1-\frac{\alpha}{2}}u(s)|^2 + |A^{\frac{1}{2}-\frac{\alpha}{2}}u'(s)|^2)ds \leq C_4 E. \tag{4. 5}$$

We introduce now

$$w(t) = A^{\frac{1}{2}-\frac{\alpha}{2}}u(t).$$

The basic estimate (4.2) applied to w instead of u gives

$$\int_s^t |A^{\frac{1}{2}}u'(\sigma)|^2 d\sigma = \int_s^t |A^{\frac{\alpha}{2}}w'(\sigma)|^2 d\sigma \leq C_5(|A^{1-\frac{\alpha}{2}}u(s)|^2 + |A^{\frac{1}{2}-\frac{\alpha}{2}}u'(s)|^2).$$

By integrating on $(0, t)$ and using (4.5) we obtain

$$\int_0^t \int_s^t |A^{\frac{1}{2}}u'(\sigma)|^2 d\sigma ds \leq C_5 C_4 E$$

which by Fubini's theorem reduces to

$$\int_0^t \sigma |A^{\frac{1}{2}}u'(\sigma)|^2 d\sigma \leq C_5 C_4 E.$$

Hence for some $K > 0$

$$\forall t > 0, \quad \int_0^t s|A^{\frac{1}{2}}u'(s)|^2 ds \leq KE. \tag{4. 6}$$

We shall in fact establish for some $M > 0$

$$\forall t > 0, \quad \int_0^t s(|A^{\frac{1}{2}}u'(s)|^2 + |u''(s)|^2)ds \leq ME. \tag{4. 7}$$

To this end, first we observe that as a consequence of (4.6)

$$\inf_{t\le s\le 2t} s^2|A^{\frac12}u'(s)|^2 \le 2t \inf_{t\le s\le 2t} s|A^{\frac12}u'(s)|^2 \le 2KE.$$

Now we choose $\tau \in (t, 2t)$ for which $u'(\tau) \in D(A^{1/2})$ with

$$\tau^2|A^{\frac12}u'(\tau)|^2 \le 2KE$$

and we integrate on $(0,\tau)$ after taking the inner product by $su''(s)$. We find

$$\int_0^\tau s|u''|^2 ds = -\int_0^\tau [s(Au,u'') + s(Bu',u'')]ds.$$

First we have

$$-\int_0^\tau s(Bu',u'')ds = -\frac12\int_0^\tau s(Bu',u')'ds = -[\frac{s}{2}(Bu',u')]_0^\tau + \int_0^\tau \frac12(Bu',u')ds \le \frac12 E.$$

Then

$$-\int_0^\tau s(Au,u'')ds = -\int_0^\tau (sAu,(u')')ds = -(sAu,u')_0^\tau + \int_0^\tau ((sAu)',u')ds$$

$$\le \tau|A^{\frac12}u(\tau)||A^{\frac12}u'(\tau)| + \int_0^\tau (Au,u')ds + \int_0^\tau s|A^{\frac12}u'|^2 ds$$

$$\le \tau|A^{\frac12}u(\tau)||A^{\frac12}u'(\tau)| + \frac12|A^{\frac12}u(\tau)|^2 + \int_0^\tau s|A^{\frac12}u'|^2 ds \le K'E.$$

By adding we find

$$\int_0^\tau s|u''|^2 ds \le K''E$$

and together with (4.6) this provides (4.7) with t replaced by τ. Since $t \le \tau$ we obtain (4.7). Finally since the function

$$t \to (|A^{\frac12}u'(t)|^2 + |u''(t)|^2)$$

is nonincreasing, we have

$$(4.\ 8)\quad \forall t>0, \quad 2\int_0^t s(|A^{\frac12}u'(s)|^2 + |u''(s)|^2)ds \ge t^2(|A^{\frac12}u'(t)|^2 + |u''(t)|^2).$$

By combining (4.7) and (4.8), we obtain (4.1) □

Theorem 4.2. *For any $\alpha < 1/2$, the semi-group $\mathcal{S}(t)$ generated on $\mathcal{H}$ by $\mathcal{A}$ satisfies*

$$\forall t>0, \quad S(t)U^0 \in D(\mathcal{A})$$

and

$$(4.\ 9)\qquad \forall t>0, \quad \|\mathcal{A}\mathcal{S}(t)U^0\|_{\mathcal{H}} \le \frac{C}{t^\beta}\|U^0\|_{\mathcal{H}}$$

with $\beta = \frac{1}{2\alpha}$. In addition if A is unbounded with A^{-1} compact, (4.9) is not satisfied for any $\beta > \frac{1}{2\alpha}$. In particular in this case the semi-group $\mathcal{S}(t)$ is not analytic.

Proof. The beginning of proof of Theorem 4.1 applies until formula (4.3). Then taking the inner product in H of (2.1) by $A^\alpha u(s)$ and integrating we find

$$\int_0^t |A^{\frac{1+\alpha}{2}}u(s)|^2 ds = -\int_0^t (u''+Bu', A^\alpha u) ds.$$

Now we have since $\alpha \le 1/2$

$$\int_0^t (Bu', A^\alpha u) ds = c\int_0^t (A^\alpha u', A^\alpha u) ds = \frac{c}{2}(|A^{\alpha/2}u(t)|^2 - |A^{\alpha/2}u(0)|^2) \ge -C_1 E.$$

Next , integrating by parts we find

$$\int_0^t (u'', A^\alpha u) ds = [(u', A^\alpha u)]_0^t - \int_0^t (u', A^\alpha u') ds$$

hence since $\alpha \le 1/2$ and by using (4.3)

$$-\int_0^t (u'', A^\alpha u) ds \le C_2 E$$

therefore

$$\int_0^t |A^{\frac{1+\alpha}{2}}u(s)|^2 ds \le (C_1 + C_2) = C_3 E. \tag{4. 10}$$

Combining (4.10) with (4.3) we deduce

$$\int_0^t (|A^{\frac{1}{2}+\frac{\alpha}{2}}u(s)|^2 + |A^{\frac{\alpha}{2}}u'(s)|^2) ds \le C_4 E \tag{4. 11}$$

from which we deduce

$$|A^{\frac{1}{2}+\frac{\alpha}{2}}u(t)| + |A^{\frac{\alpha}{2}}u'(t)| \le \frac{C}{\sqrt{t}}(|A^{\frac{1}{2}}u(0)| + |A^0 u'(0)|). \tag{4. 12}$$

Since time-translation and multiplication by A^α commutates with the equation, replacing t by $\frac{t}{n}$ and iterating (4.12) n times we deduce easily

$$|A^{\frac{1}{2}+\frac{n\alpha}{2}}u(t)| + |A^{\frac{n\alpha}{2}}u'(t)| \le \frac{C(n)}{(\sqrt{t})^n}(|A^{\frac{1}{2}}u(0)| + |A^0 u'(0)|). \tag{4. 13}$$

Now (4.13) is valid for integer values of n and by interpolation, we extend it easily for all real $n > 0$. Finally choosing $n = \frac{1}{\alpha}$ we obtain (4.9).

In order to prove the optimality result, we set $p = \frac{1}{\alpha}$, we select $\gamma > 0$ and for any $\lambda > 0$ with $\gamma\lambda^p - \lambda^2 > 0$ we set

$$\omega := \sqrt{\gamma\lambda^p - \lambda^2}; \quad y(t) = y_\lambda(t) = e^{-\lambda t}\cos\omega t.$$

Then

$$y'(t) = -\lambda e^{-\lambda t}\cos\omega t - \omega e^{-\lambda t}\sin\omega t$$
$$y''(t) = \lambda^2 e^{-\lambda t}\cos\omega t + 2\omega\lambda e^{-\lambda t}\sin\omega t - \omega^2 e^{-\lambda t}\cos\omega t$$

and

$$y''(t) + 2\lambda y'(t) = -(\lambda^2 + \omega^2) e^{-\lambda t}\cos\omega t$$

so that y is a solution of

$$y'' + \gamma\lambda^p y + 2\lambda y' = 0 \tag{4. 14}$$

which satisfies

$$\lambda^p y^2(0) + y'^2(0) = \lambda^p + \lambda^2.$$

We are interested in the behavior of the energy for *large* values of λ and *small* values of t when $\gamma \geq \gamma_0 > 0$. We select

$$t := t(\lambda) = \pi \frac{[\frac{\omega}{\lambda}] + 1/2}{\omega}$$

where $[\frac{\omega}{\lambda}]$ denotes the integer part of $\frac{\omega}{\lambda}$. As $\lambda \to \infty$ we have

$$t(\lambda) \sim \frac{\pi}{\lambda}$$

and it follows

$$\lambda^p {y'}^2(t(\lambda)) \sim \lambda^p \omega^2 e^{-2\pi}$$

$$t^p(\lambda)\lambda^p {y'}^2(t(\lambda)) \sim \pi^p \omega^2 e^{-2\pi} \sim \pi^p e^{-2\pi} \gamma \lambda^p$$

In particular for any $\varepsilon > 0$ we have

$$\lim_{\lambda \to \infty} (t(\lambda))^{p-\varepsilon} \quad \frac{\lambda^p {y'_\lambda}^2(t(\lambda)) + {y''_\lambda}^2(t(\lambda))}{\lambda^p y^2(0) + {y'}^2(0)} = \infty$$

uniformly for $\gamma \geq \gamma_0 > 0$.

Finally let μ be a large eigenvalue of A with associated eigenfunction ϕ_μ and set

$$\gamma = \mu (c/2)^{-p}; \quad \lambda = (\frac{\mu}{\gamma})^{1/p} \Longleftrightarrow \gamma \lambda^p = \mu.$$

Since $y = y(\lambda)$ is a solution of

$$y'' + \mu y + 2\gamma^{-1/p}\mu^{1/p} y' = y'' + \mu y + c\mu^\alpha y' = 0.$$

It is now easy to see that $u(t) := y(t)\phi_\mu$ is a solution of (2.1) which does not satisfy (4.4) for any $\beta > \frac{1}{2\alpha}$. □

Theorems 4.1 and 4.2 imply a stronger time-smoothing effect property. More precisely we have

Theorem 4.3. *For any $\alpha > 0$, and for any solution u of*

$$u'' + Au + cA^\alpha u' = 0$$

we have

$$\forall \delta > 0, u \in C^\infty([\delta, \infty), V))$$

In addition the operator

$$(u(0), u'(0)) \in V \times H \to u^{(k)} \in L^\infty[\delta, \infty), V)$$

is bounded for each fixed value of k .

5. Existence of a spatial smoothing effect

Combining the result of Theorem 4.3 and the inclusion $D(\mathcal{A}) \subset H^\gamma \times V$ obtained in Theorem 3.1 for $0 < \alpha < 1$, by an easy induction argument we obtain

Theorem 5.1. *Assuming $\alpha < 1$, for any solution u of*

$$u'' + Au + cA^\alpha u' = 0$$

we have

$$\forall n \in N, \quad \forall \delta > 0, u \in C^\infty([\delta, \infty), D(A^n))$$

In addition the operator

$$(u(0), u'(0)) \in V \times H \to u^{(k)} \in L^\infty([\delta, \infty), D(A^n))$$

is bounded for each fixed value of k and n.

Remark. If on the other hand $\alpha \geq 1$ there is no spatial smoothing effect anymore. For instance if we consider the special case

$$\Omega = (0, \pi) \quad H = L^2(\Omega) \quad V = H_0^1(\Omega) \quad A = -\Delta, \quad B = 2A$$

we have special solutions of the form

$$u(t, x) = \sum a_n e^{\left(-n^2 + \sqrt{n^2(n^2-1)}\right)t} \sin nx$$

with

$$u(0) = \sum a_n \sin nx$$

and

$$\|u(0)\|^2_{D(A^s)} = \sum n^{4s} a_n^2$$

$$\|u(t)\|^2_{D(A^s)} = \sum n^{4s} a_n^2 e^{\left(-n^2 + \sqrt{n^2(n^2-1)}\right)2t}.$$

Since

$$-n^2 + \sqrt{n^2(n^2-1)} = \frac{-n^2}{n^2 + \sqrt{n^2(n^2-1)}} \geq -1$$

we find

$$\|u(0)\|^2_{D(A^s)} \leq \|u(t)\|^2_{D(A^s)} e^{2t}$$

This formal estimate can be easily worked out to show that if $u(t) \in D(A^s)$ for some $t > 0$, we must have $u(0) \in D(A^s)$. Hence there is no spatial smoothing effect whatsoever in such a situation.

6. Regularity: conservation properties

Let $L = L^* \geq 0$ on H and assume that there is a second Banach space X such that

$$\bigcap_{n \geq 1} D(L^n) \subset X \subset H \tag{6. 1}$$

with dense imbeddings. The norm in X is denoted by $\| \, \|$ and we assume that $\exp(-tL)$ is a C^0 semigroup of bounded operators on X.

First we consider the problem

$$u'' + L^2 u + cLu' = 0 \tag{6. 2}$$

Then we have

Theorem 6.1. *Let $c \geq 2$, and let L be coercive on X. Introducing*

$$D_X(L) = \{x \in D(L), \, Lx \in X\}$$

assume that the set $\{x \in D_X(L), \quad \|Lx\| \leq 1\}$ is closed in X and that $\exp(-tL)$ is an analytic semigroup of bounded operators on X. Let u be the unique solution of (6.2) satisfying

$$u(0) = u^0 \in D_X(L), \quad u'(0) = u^1 \in X$$

whose existence is insured by Theorem 2.1 with $A = L^2$ and $B = cL$. Then we have

$$\forall t > 0, \quad u(t) \in D_X(L), \quad u'(t) \in X$$

with

$$\sup_{t>0}\{\|u(t)\| + \|Lu(t)\| + \|u'(t)\|\} \le C(\|u^0\| + \|Lu^0\| + \|u^1\|) \tag{6. 3}$$

Proof. We start with

$$(u^0, u^1) \in (\bigcap_{n\ge1} D(L^n))^2$$

The main idea is to look for $\alpha, \beta > 0$ such that

$$u'' + L^2u + cLu' = (u' + \alpha Lu)' + \beta L(u' + \alpha Lu).$$

This identity reduces to the system

$$\alpha + \beta = c; \quad \alpha\beta = 1$$

which has real solutions (α, β) as a consequence of the assumption $c \ge 2$. Then since $v = u' + \alpha Lu$ is a solution of

$$v' + \beta Lv = 0$$

we have (cf. e.g. [6]) with $K = \|u^1\| + \alpha\|Lu^0\|$

$$\forall t > 0, \quad \|v(t)\|_{D_X(L^{1/2})} \le C_1 t^{-1/2}\|u^1 + \alpha Lu^0\| \le C_1 t^{-1/2} K.$$

Then we have

$$u(t) = e^{-\alpha Lt}u^0 + \int_0^t e^{-\alpha L(t-s)}v(s)ds$$

hence by [6]

$$\forall t > 0, \quad \|u(t)\|_{D_X(L)} \le C_2\|u^0\|_{D_X(L)} + C_3\int_0^t C_1(t-s)^{-1/2}s^{-1/2}Kds.$$

This provides the bound on $\|u(t)\| + \|Lu(t)\|$. Finally since

$$\forall t > 0, \quad \|v(t)\| \le K$$

and

$$u'(t) = v(t) - \alpha Lu(t)$$

the estimate on $\|u'(t)\|$ follows easily. Then the general case

$$(u^0, u^1) \in D_X(L) \times X$$

follows by a density argument. $\square$

We next consider the problem

$$u'' + Lu + cLu' = 0. \tag{6. 4}$$

We assume that $\exp(-tL)$ is a C^0 semigroup of bounded operators on X. Then we have

Theorem 6.2. *Under the condition (6.1), Let u be the unique solution of (6.4) satisfying*

$$u(0) = u^0 \in X, \quad u'(0) = u^1 \in X$$

whose existence is insured by Theorem 2.1 with $A = L$ and $B = cL$. Then we have

$$\forall t > 0, \quad u(t) \in X \quad u'(t) \in X$$

If in addition we assume

$$\| \exp(-tL) \|_{\mathcal{L}(X)} \leq M e^{-\lambda t} \tag{6. 5}$$

for some $M > 1, \lambda > 0$. such that

$$c^2 \lambda > M + 1 \tag{6. 6}$$

then (u, u') is bounded for $t \geq 0$ and

$$\sup_{t>0} \{ \|u(t)\| + \|u'(t)\| \} \leq C(c, M, \lambda)(\|u^0\| + \|u^1\|) \tag{6. 7}$$

.

Proof. As before we consider first

$$(u^0, u^1) \in (\bigcap_{n \geq 1} D(L^n))^2$$

The main idea is to introduce

$$v := u' + \frac{1}{c} u$$

so that

$$v' + cLv = u'' + \frac{1}{c} u' + cLu' + Lu = \frac{1}{c} u' = \frac{1}{c}(v - \frac{1}{c} u)$$

Let J be any closed subinterval of $[0, +\infty)$. By setting

$$\|u\|_{J,\infty} := \sup_{t \in J} \|u(t)\|$$

and similarly

$$\|v\|_{J,\infty} := \sup_{t \in J} \|v(t)\|$$

from

$$v' + cLv - \frac{1}{c} v = -\frac{1}{c^2} u$$

we obtain

$$\forall t \in J, \quad \|v(t)\| \leq M \exp(\frac{1}{c} t) \|v^0\| + \frac{M}{c^2} \exp(\frac{1}{c} t) |J| \|u\|_{J,\infty} \leq C(\|v^0\| + |J| \|u\|_{J,\infty})$$

for $|J| \leq 1$ and

$$\forall t \in J, \quad \|u(t)\| \leq \|u^0\| + c\|v\|_{J,\infty}$$

The result easily follows locally by selecting $|J|$ small enough, for instance $Cc|J| \leq \frac{1}{2}$ and using a density argument. Then a simple induction argument concludes the proof, since the condition on $|J|$ is independent of the initial data and the equation is autonomous.

When (6.5)-(6.6) are assumed, the same calculation for an arbitray J now gives

$$\forall t \in J, \quad \|v(t)\| \le M \exp[-(\lambda c - \frac{1}{c})t]\|v_0\| + \frac{M}{\lambda c^3 - c}\|u\|_\infty$$

and

$$\forall t \in J, \quad \|u(t)\| \le \|u^0\| + c\|v\|_\infty$$

Hence if $\lambda c^2 > 1$ we find

$$\|v\|_\infty \le M\|v_0\| + \frac{M}{\lambda c^3 - c}\|u\|_\infty \le M\|v_0\| + \frac{M}{\lambda c^3 - c}(\|u^0\| + c\|v\|_\infty)$$

which gives the desired bound for v if

$$\frac{M}{\lambda c^2 - 1} < 1$$

a condition equivalent to (6.6). Then the bound on u follows automatically. This bound is valid on each interval $J = [0, T]$ and the result (6.7) follows. The general case is obtained by density. $\square$

7. Main examples

Example 7.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$. We consider the problem*

$$(7.\ 1) \qquad \begin{cases} u_{tt} - \Delta u - c\Delta u_t = 0 \quad on \quad \mathbb{R}^+ \times \Omega \\ u = \Delta u = 0 \quad on \quad \mathbb{R}^+ \times \partial\Omega \\ u(0) = u^0 \quad u'(0) = u^1 \end{cases}$$

We set

$$H = L^2(\Omega) \quad ; \quad V = H_0^1(\Omega)$$

$$X = C_0(\Omega) = \{u \in C(\overline{\Omega}), \quad u = 0 \ on\ \partial\Omega\}$$

For any $(u^0, u^1) \in V \times H$, the unique mild solution u of (7.1) belongs to $C^\infty((0,\infty), V)$. If in addition we assume $(u^0, u^1) \in X \times X$, then for all $t \ge 0$ $(u(t,.), u_t(t,.)) \in X \times X$. For c large enough we have the stronger property $u \in W^{1,\infty}((0,\infty), X)$.The same result is also valid if X is replaced by $X_p = L^p(\Omega)$ for any $p \in [2, \infty)$.

Example 7.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$. We consider the problem*

$$(7.\ 2) \qquad \begin{cases} u_{tt} + \Delta^2 u - c\Delta u_t = 0 \quad on \quad \mathbb{R}^+ \times \Omega \\ u = \Delta u = 0 \quad on \quad \mathbb{R}^+ \times \partial\Omega \\ u(0) = u^0 \quad u'(0) = u^1 \end{cases}$$

For any $(u^0, u^1) \in$ We set

$$H = L^2(\Omega) \quad ; \quad V = \{u \in H_0^1(\Omega), \Delta u \in H\}$$

$$X = C_0(\Omega) = \{u \in C(\overline{\Omega}), \quad u = 0 \ on\ \partial\Omega\}$$

For any $(u^0, u^1) \in V \times H$, the unique mild solution u of (7.1) belongs to $C^\infty((0,\infty), V) \cap C^\infty((0,\infty) \times \Omega)$. If in addition we assume $c \ge 2$ and $(u^0, u^1) \in X \times X$, then $u \in L^\infty((0,\infty), X)$. Finally if $c \ge 2$, $(u^0, u^1) \in X \times X$

and $\Delta^2 u^0 \in X$*, then* $\Delta^2 u \in L^\infty((0,\infty), X)$*;* $\quad u_t \in L^\infty((0,\infty), X)$*. The same result is also valid if* X *is replaced by* $X_p = L^p(\Omega)$ *for any* $p \in [2,\infty)$*.*

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